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Kac's question, planar isospectral pairs and involutions in projective space: II. Classification of generalized projective isospectral data

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Abstract

In *Am. Math. Monthly* (**73** 1–23 (1966)), Kac asked his famous question 'Can one hear the shape of a drum?', which was eventually answered negatively in Gordon *et al* (1992 *Invent. Math.* **110** 1–22) by construction of planar isospectral pairs. Giraud (2005 *J. Phys. A: Math. Gen.* **38** L477–83) observed that most of the known examples can be generated from solutions of a certain equation which involves a set of involutions of an n-dimensional projective space over some finite field. He then generated all possible solutions for $n = 2$, when the involutions fix the same number of points. In Thas (2006 *J. Phys. A: Math. Gen.* **39** L385–8) we showed that no other examples arise for any other dimension, still assuming that the involutions fix the same number of points. In this paper we study the problem for involutions not necessarily fixing the same number of points, and solve the problem completely.

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1. 'Can one hear the shape of a drum?': setting

This paper is a sequel to the letter [8]. In [6], M Kac formulated his famous question 'Can one hear the shape of a drum?', which amounts to finding non-isometric planar simply connected domains for which the sets $\{E_n \parallel n \in \mathbb{N}\}$ of solutions of the stationary Schrödinger equation

$$(\Delta + \mathbf{E})\Psi = 0 \quad \text{with} \quad \Psi_{|\text{Boundary}} = 0$$

are identical.

Any example of such a pair of non-congruent planar isospectral domains yields a counter example to Kac's question. Several counter examples were constructed to the analogous

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question on Riemannian manifolds (cf Brooks [1]), but for Euclidian domains the question appeared to be much harder. Finally, Gordon, Webb and Wolpert provided a pair of simply connected non-isometric Euclidian isospectral domains—also called 'planar isospectral pairs' or 'isospectral billiards'—in [4]. Other examples were found later, see for instance the paper by Buser, Conway, Doyle and Semmler [2]. Essentially only a finite number of planar isospectral pairs are known at present.

In [3], Giraud studied an equation that yields candidates for generating isospectral billiards, associated with a set of involutory automorphisms of an n-dimensional projective space over a finite field which fix the same number of points of the space. For $n = 2$, he managed to solve the equation, and generated by computer all isospectral billiards that can arise from the solutions. In [8] the author of the present paper solved the general equation, for any dimension, and showed that it essentially reduced to the planar case.

In this paper, we handle the equation for sets of involutions that do not necessarily fix the same number of points, thus solving Giraud's problem in its entirity. The main result reads as follows (definitions can be found in the next section).

Theorem 1.1. *Let* $\mathbf{P} = \mathbf{PG}(n, q)$ *be the* n*-dimensional projective space over the finite field* $\mathbf{GF}(q)$, $n \geqslant 2$, *and suppose there exists generalized projective isospectral data* $(\mathbf{P}, \{\theta^{(i)}\}, r)$ *which yield isospectral billiards. Then either* $n = 2$, the $\theta^{(i)}$ *fix the same number of points of* $\mathbf{P}$*, and the solutions are as described in [3], or* $n = 3, r = 3$ *and* $q = 2$*, and again the examples can be found in [3].*

2. Projective and generalized projective isospectral data

For a pair of isospectral billiards on N copies of a tile with r sides (so that one may take r to be at least 3), one needs r involutions acting on a set of N letters, with the property that the graph formed by joining each pair of points that are interchanged by at least one of the involutions, has no loops and is connected. For any involution the number of edges is $(N - s)/2$, with s the number of fixed points of the involution. The total number of edges must equal $N - 1$, and the group of transformations generated by the involutions must act transitively on the set of N points.

In [3], Giraud studies triples $(\mathbf{P}, \{\theta^{(i)}\}, r)$, where $\mathbf{P}$ is a finite projective space of dimension at least 2 [5], and $\{\theta^{(i)}\}$ a set of r nontrivial involutory automorphisms of $\mathbf{P}$, satisfying the following equation

$$r(|\mathbf{P}| - \mathrm{Fix}(\theta)) = 2(|\mathbf{P}| - 1), \tag{1}$$

for some natural number $r \geqslant 3$, where $\mathrm{Fix}(\theta) = \mathrm{Fix}(\theta^{(i)})$ is a constant number of fixed points of $\mathbf{P}$ under each $\theta^{(i)}$, and $|\mathbf{P}|$ is the number of points of $\mathbf{P}$. In [8], we called such a triple $(\mathbf{P}, \{\theta^{(i)}\}, r)$ *projective isospectral data.*

For projective spaces of dimension 2 Giraud determined all solutions of equation (1).

Theorem 2.1 (Giraud [3])**.** *Let* $\mathbf{P} = \mathbf{PG}(2, q)$ *be the two-dimensional projective space over the finite field* $\mathbf{GF}(q)$*, and suppose there exists projective isospectral data* $(\mathbf{P}, \{\theta^{(i)}\}, r)$*. If* q *is not a square, then* $(r, q) \in \{(3, 2), (3, 3)\}$*. If* q *is a square, then there are no integer solutions of equation (1).*

He then showed that all examples of [2] can be generated by computer from the obtained data. In [8] we proved the next generalization.

Theorem 2.2 (Thas [8])**.** *Let* $\mathbf{P} = \mathbf{PG}(n, q)$ *be the* n*-dimensional projective space over the finite field* $\mathbf{GF}(q)$*, and suppose there exists projective isospectral data* $(\mathbf{P}, \{\theta^{(i)}\}, r)$*. Then* q

cannot be a square. If q is not a square, then $(r, n, q) \in \{(3, 2, 2), (3, n, 3)\}$, where in the case $(r, n, q) = (3, n, 3)$ each $\theta^{(i)}$ fixes pointwise a hyperplane, and also a point not in that hyperplane. However, this class of solutions only generates planar isospectral pairs if $n = 2$.

Call a triple $(\mathbf{P}, \{\theta^{(i)}\}, r)$, where $\mathbf{P}$ is a finite projective space of dimension at least 2, and $\{\theta^{(i)}\}$ a set of r nontrivial involutory automorphisms of $\mathbf{P}$, satisfying

$$r(|\mathbf{P}|) - \sum_{j=1}^{r} \mathrm{Fix}(\theta^{(j)}) = 2(|\mathbf{P}| - 1), \tag{2}$$

for some natural number $r \geqslant 3$, 'generalized projective isospectral data'.

3. Involutions in finite projective space

Let $\mathbf{PG}(n, q)$, $n \in \mathbb{N} \cup \{-1\}$, be the n-dimensional projective space over the Galois field $\mathbf{GF}(q)$ with q elements, so that q is a prime power; we have $|\mathbf{PG}(n, q)| = \frac{q^{n+1}-1}{q-1}$. (Note that $\mathbf{PG}(-1, q)$ is the empty space.)

We discuss the different types of involutions that can occur in the automorphism group of a finite projective space [7].

- *Baer involutions*. A *Baer involution* is an involution which is not contained in the linear automorphism group of the space, so that q is a square, and it fixes an n-dimensional subspace over $\mathbf{GF}(\sqrt{q})$ pointwise.
- *Linear involutions in even characteristic*. If q is even, and θ is an involution which is not of Baer type, θ must fix an m-dimensional subspace of $\mathbf{PG}(n, q)$ pointwise, with $1 \leqslant m \leqslant n \leqslant 2m + 1$. In fact, to avoid trivialities, one assumes that $m \leqslant n - 1$.
- *Linear involutions in odd characteristic*. If θ is a linear involution of $\mathbf{PG}(n, q)$, q odd, the set of fixed points is the union of two disjoint complementary subspaces. Denote these by $\mathbf{PG}(k, q)$ and $\mathbf{PG}(n - k - 1, q)$, $k \geqslant n - k - 1 > -1$.

4. Classification of planar isospectral pairs with generalized projective isospectral data

We distinguish four cases, according to the characteristic of $\mathbf{GF}(q)$, and whether q is a square or not. The dimension of $\mathbf{P}$ is always at least 2. In each of the cases, the aim is to show that equation (2) has no 'appropriate' solutions.

Even characteristic

4.1. $\{\theta^{(i)}\}$ contains Baer involutions

It is convenient to denote $\mathbf{P}$ by $\mathbf{PG}(n, q^2)$ in this case. Equation (2) becomes

$$r\frac{q^{2(n+1)} - 1}{q^2 - 1} - \sum_{j=1}^{i} \frac{q^{n+1} - 1}{q - 1} - \sum_{j=1}^{r-i} \frac{q^{2(m_j+1)} - 1}{q^2 - 1} = 2\left(\frac{q^{2(n+1)} - 1}{q^2 - 1} - 1\right),$$

with m_j integers satisfying $1 \leqslant m_j \leqslant n \leqslant 2m_j + 1$, and $0 < i < r$. So

$$r(q^{2(n+1)} - 1) - i(q^{n+1} - 1)(q + 1) - \sum_{j=1}^{r-i} \left(q^{2(m_j+1)} - 1\right) = 2(q^{2(n+1)} - q^2).$$

Since the right-hand side is divisible by q^2, it follows that q^2 divides $-r+iq+i+(r-i) = iq$, so $q|i < r$. We obtain

$$r(q^{2(n+1)}-1) - i(q^{n+1}-1)(q+1) - \sum_{j=1}^{r-i}\left(q^{2(m_j+1)}-1\right) \geqslant q^{2n+3} - q - q(q^{2n}-1)$$
$$= q^{2n+3} - q^{2n+1}.$$

If $q \geqslant 4$, we deduce

$$q^{2n+3} - q^{2n+1} > 3q^{2(n+1)} > 2q^{2(n+1)} - 2q^2$$

contradiction. It remains to handle the case $q = 2$. So we have to look at solving

$$r(2^{2(n+1)}-1) - i(2^{n+1}-1)3 - \sum_{j=1}^{r-i}\left(2^{2(m_j+1)}-1\right) = 2^{2n+3} - 8. \tag{3}$$

First, we observe that 4 divides i, so that $r > i \geqslant 4$. It will be sufficient to prove the next inequality

$$(r-2)2^{2n-1} - 3 \times 2^{n-2}i + \frac{i}{4} - \sum_{j=1}^{r-i} 2^{2m_j-1} + 1 > 0.$$

Under our assumptions, the latter expression follows from the following facts:

$$\begin{cases} (r-i)2^{2n-1} \geqslant \sum_{j=1}^{r-i} 2^{2m_j-1} & (\text{as } m_j \leqslant n) \\ (i-2)2^{2n-1} \geqslant 3 \times 2^{n-2}i & (\text{as } i \geqslant 4 \text{ and } n \geqslant 2) \\ 1 + \frac{i}{4} > 0. \end{cases}$$

4.2. q is even and $\{\theta^{(i)}\}$ *does not contain Baer involutions*

We have to consider the equation

$$r\frac{q^{n+1}-1}{q-1} - \sum_{j=1}^{r}\frac{q^{m_j+1}-1}{q-1} = 2\left(\frac{q^{n+1}-1}{q-1} - 1\right),$$

with m_j integers which are not all equal, and satisfying $1 \leqslant m_j \leqslant n \leqslant 2m_j + 1$. So

$$r(q^{n+1}-1) - \sum_{j=1}^{r}\left(q^{m_j+1}-1\right) = 2(q^{n+1}-q).$$

We know that $\mathrm{Max}\{m_1, m_2, \ldots, m_r\} \leqslant n-1$, so the left-hand side is strictly larger than $rq^{n+1} - rq^n$. Clearly, if $q \geqslant 4$, then

$$rq^{n+1} - rq^n \geqslant 3q^{n+1} - 3q^n > 2q^{n+1} - 2q.$$

Put $q = 2$, and observe that $r = 3$ is the only possible value for r we have to consider. So we have to solve

$$3.2^{n+1} - \sum_{j=1}^{3} 2^{m_j+1} = 2(2^{n+1}-2).$$

As 4 is the largest power of 2 that divides the right-hand side, there is a $\lambda \in \{1, 2, 3\}$ for which $m_\lambda = 1$, say m_1 (note that if $m_j = 0$, **P** has dimension 1). This means that the fixed points space of $\theta^{(1)}$ in $\mathbf{PG}(n, q)$ is a line, implying that $n = 3$ (the planar case follows from [3]—since q is not a square, all involutions of $\mathbf{PG}(2, q)$ are linear and fix the same number of points). One easily concludes that $m_2 = m_3 = 2$ yields a solution. However, Giraud generated all possible isospectral billiards arising from (generalized) projective isospectral data in $\mathbf{PG}(3, 2)$ by computer, cf [3].

Odd characteristic

We start with a lemma.

Lemma 4.1. *Let ϕ be a linear involution of the projective space $\mathbf{PG}(l, q)$, q odd, $l \geqslant 2$. Then the number of fixed points is at most $\frac{q^l+q-2}{q-1}$.*

Proof. We know that ϕ fixes pointwise two complementary subspaces $\mathbf{PG}(k, q)$ and $\mathbf{PG}(l-k-1, q)$, so

$$\text{number of fixed points} = \frac{q^{k+1} + q^{l-k} - 2}{q-1}.$$

Clearly, one obtains a maximal solution when one of the fixed points spaces is a hyperplane. □

Lemma 4.1 will be used without further reference.

4.3. $\{\theta^{(i)}\}$ contains Baer involutions

It is again convenient to denote $\mathbf{P}$ by $\mathbf{PG}(n, q^2)$. The equation becomes

$$r\frac{q^{2(n+1)}-1}{q^2-1} - \sum_{j=1}^{i}\frac{q^{n+1}-1}{q-1} - \sum_{j=1}^{r-i}\frac{q^{2(m_j+1)}-1}{q^2-1} - \sum_{j=1}^{r-i}\frac{q^{2(n-m_j)}-1}{q^2-1} = 2\left(\frac{q^{2(n+1)}-1}{q^2-1} - 1\right),$$

with m_j integers satisfying $m_j \geqslant n - m_j - 1$, and $0 \neq i \neq r$. Multiplying with $q^2 - 1$, we obtain

$$\begin{aligned} r(q^{2(n+1)}-1) &- \sum_{j=1}^{i}(q^{n+1}-1)(q+1) - \sum_{j=1}^{r-i}(q^{2(m_j+1)}-1) - \sum_{j=1}^{r-i}(q^{2(n-m_j)}-1) \\ &\geqslant r(q^{2(n+1)}-1) - i(q^{n+1}-1)(q+1) - (r-i)(q^{2n}+q^2-2) \\ &\geqslant 3(q^{2(n+1)}-1) - 2(q^{2n}+q^2-2) - (q^{n+1}-1)(q+1). \end{aligned}$$

Now the latter expression equals

$$2q^{2(n+1)} + [q^{2(n+1)} - 2q^{2n} - q^{n+2} - q^{n+1} - 2q^2 + q + 2] > 2q^{2(n+1)} > 2q^{2(n+1)} - 2q^2.$$

4.4. q is odd and $\{\theta^{(i)}\}$ does not contain Baer involutions

The equation becomes

$$r\frac{q^{n+1}-1}{q-1} - \sum_{j=1}^{r}\frac{q^{m_j+1}-1}{q-1} - \sum_{j=1}^{r}\frac{q^{n-m_j}-1}{q-1} = 2\left(\frac{q^{n+1}-1}{q-1} - 1\right),$$

with m_j integers satisfying $m_j \geqslant n - m_j - 1$, and which are not all equal. The left-hand side is strictly larger than

$$r(q^{n+1}-1) - r(q^n+q-2) = rq^{n+1} - rq^n - rq + r \geqslant 3q^{n+1} - 3q^n - 3q + 3,$$

using that $r \geqslant 3$, and

$$3q^{n+1} - 3q^n - 3q + 3 \geqslant 2q^{n+1} - 2q \quad \Leftrightarrow \quad q^{n+1} \geqslant 3q^n + q - 3,$$

which is 'always' the case.

Together with the main result of [8], we have shown the next theorem.

Theorem 4.2. *Let* $\mathbf{P} = \mathbf{PG}(n, q)$ *be the n-dimensional projective space over the finite field* $\mathbf{GF}(q)$, $n \geqslant 2$, *and suppose there exists generalized projective isospectral data* $(\mathbf{P}, \{\theta^{(i)}\}, r)$ *which yield isospectral billiards. Then either* $n = 2$, *the* $\theta^{(i)}$ *fix the same number of points of* $\mathbf{P}$, *and the solutions are as described in [3], or* $n = 3, r = 3$ *and* $q = 2$, *and again the examples can be found in [3].*

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